

ON THE GROTHENDIECK–SERRE CONJECTURE ON PRINCIPAL BUNDLES IN MIXED CHARACTERISTIC

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ABSTRACT. Let R be a regular local ring. Let \mathbf{G} be a reductive R -group scheme. A conjecture of Grothendieck and Serre predicts that a principal \mathbf{G} -bundle over R is trivial if it is trivial over the quotient field of R . The conjecture is known when R contains a field. We prove the conjecture for a large class of regular local rings *not* containing fields in the case when \mathbf{G} is split.

1. INTRODUCTION

Let R be a regular local ring; let \mathbf{G} be a reductive group scheme over R . A conjecture of Grothendieck and Serre (see [Ser, Remarque, p.31], [Gro2, Remarque 3, p.26–27], and [Gro5, Remarque 1.11.a]) predicts that a principal \mathbf{G} -bundle over R is trivial, if it is trivial over the fraction field of R . Recently this has been proved in the case when R contains a field: in [FP] if the field is infinite, in [Pan3] if the field is finite. In this paper we consider the case when R contains no field, that is, the case of *mixed characteristic*.

Note that a regular local ring R contains no field if and only if there is a prime number p (necessarily unique) such that p is neither invertible nor zero in R . In this case R contains the localization \mathbb{Z}_p of \mathbb{Z} at the prime ideal $p\mathbb{Z}$.

Thus, we assume that R is a \mathbb{Z}_p -algebra. *We will assume that R is a regular \mathbb{Z}_p -algebra or, equivalently, that R/pR is a regular ring.* In this case a theorem of Popescu [Pop, Spi, Swa] reduces the question to the case, when R is a localization of a finitely generated smooth \mathbb{Z}_p -algebra A at a maximal ideal. Taking the closure of $\text{Spec } A$ in $\mathbb{P}_{\mathbb{Z}_p}^N$, we may assume that R is the local ring of a closed point x on an integral scheme X projective over \mathbb{Z}_p .

Additionally, we will assume that (I) *the fiber X_p is generically reduced, and* (II) *that the set of singular points of X intersects X_p by a subset of codimension at least two in X_p .* Note that condition (I) is satisfied if the fiber X_p is irreducible because the projection is smooth at x . On the other hand, both conditions are satisfied if the singular locus of the projection $X \rightarrow \text{Spec } \mathbb{Z}_p$ has codimension at least 3 in X .

Below we will prove the conjecture of Grothendieck and Serre under the above assumptions when the group scheme \mathbf{G} is *split*; see Theorem 1. We work in a slightly greater generality: we consider projective schemes over any excellent discrete valuation ring B , not just \mathbb{Z}_p -schemes.

We note that previously the conjecture was known in a very few mixed characteristic cases, namely, when \mathbf{G} is a torus [CTS], when $\dim R = 1$, when R is Henselian [Nis2]. Also, in [Nis3] the conjecture is proved when \mathbf{G} is quasisplit and $\dim R = 2$ but there it is assumed that the residue field of R is infinite. Thus our results are new even in dimension two.

1.1. Definitions and conventions. A group scheme \mathbf{G} over a scheme S is called *reductive* if it is affine and smooth as an S -scheme and if, moreover, all its geometric fibers are connected reductive algebraic groups. This definition of a reductive R -group scheme coincides with [DG, Exp. XIX, Def. 2.7].

A reductive group scheme \mathbf{G} over a regular local scheme S is *split* if it contains a maximal torus $\mathbf{T} \subset \mathbf{G}$ such that $\mathbf{T} \simeq (\mathbb{G}_{\mathrm{m}})_S^r$ for some r (cf. [DG, Exp. XXII, Prop. 2.2]).

Let \mathbf{G} be a scheme faithfully flat and finitely presented over S . An S -scheme \mathcal{G} with a left action of \mathbf{G} is a *principal \mathbf{G} -bundle over S* , if \mathcal{G} is faithfully flat and finitely presented over S , and the natural morphism $\mathbf{G} \times_S \mathcal{G} \rightarrow \mathcal{G} \times_S \mathcal{G}$ is an isomorphism (see [Gro6, Sect. 6]). If T is an S -scheme, we will use the term “principal \mathbf{G} -bundle over T ” to mean a principal $\mathbf{G} \times_S T$ -bundle over T . We usually skip the adjective ‘principal’ as we are only considering principal \mathbf{G} -bundles. The pointed set of isomorphism classes of \mathbf{G} -bundles over S is denoted by $H_{\mathrm{fppf}}^1(S, \mathbf{G})$ (as every such bundle is locally trivial in the fppf topology). The subset corresponding to étale locally trivial bundles is denoted by $H_{\mathrm{\acute{e}t}}^1(S, \mathbf{G})$. We note that if \mathbf{G} is smooth over S , then we have

$$H_{\mathrm{\acute{e}t}}^1(S, \mathbf{G}) = H_{\mathrm{fppf}}^1(S, \mathbf{G}).$$

For a scheme S we denote by \mathbb{A}_S^m the m -dimensional affine space over S and by \mathbb{P}_S^m the m -dimensional projective space. We say that an S -scheme X is *projective over S* , if Zariski locally over S it admits a closed S -embedding into \mathbb{P}_S^m , where m may vary.

The symbol ‘ \simeq ’ means that two objects are isomorphic; we use the equality ‘=’ to emphasize that the isomorphism is canonical. We use boldface font for group schemes (e.g. \mathbf{G} , \mathbf{B} , etc.) and the calligraphic font for principal bundles (e.g. \mathcal{G} , \mathcal{E} , etc.).

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2. MAIN RESULTS

Fix an excellent discrete valuation ring B and a split reductive B -group scheme \mathbf{G} . Let $b \in \mathrm{Spec} B$ be the closed point. For a B -scheme X we denote by X_b its fiber over b . Let X be an integral scheme and $\pi : X \rightarrow \mathrm{Spec} B$ be a flat projective morphism. Denote by X^{sing} the set of point of X where X is singular. Then X^{sing} is closed because B is excellent. Assume that $\pi : X \rightarrow \mathrm{Spec} B$ satisfies the following properties

- (I) The special fiber X_b is generically smooth.
- (II) The intersection $X^{sing} \cap X_b$ has codimension at least two in X_b .

We note that $X^{sing} \cap X_b$ is in general smaller than the singular locus of X_b . Our main result is the following theorem.

Theorem 1. *Let $x \in X$ be a closed point such that π is smooth at x . Then a principal \mathbf{G} -bundle over $\mathcal{O}_{X,x}$ is trivial, if it has a rational section.*

For the proof, see Section 3.

Remarks 2.1.

- The Grothendieck–Serre conjecture is known for regular local rings containing finite field [Pan3, Pan1, Pan2] (see also [Pan4]). Thus we may assume that B does not contain a finite field. In this case B is automatically excellent; see [Gro4, Scholie 7.8.3(iii)].
- As explained in the introduction, we are mostly interested in the case, when B is a localization of a number field; obviously it satisfies the requirements.
- Condition (i) is satisfied if X_b is irreducible, because π is smooth at x .
- If the residue field of b is perfect, then Condition (I) is equivalent to the condition that X_b has no multiple components.
- We expect that, more generally, the theorem and its proof hold for the semi-local rings of finitely many closed points on X . Note that the conjecture is proved for split group schemes in the case of semi-local Dedekind domains in [PS], which partially extends the results of [Nis2].

The following result of independent interest will be used in the proof.

Theorem 2. *Let R be a Noetherian local ring. Let \mathbf{G} be a split reductive group scheme over R . Let \mathcal{F} be a principal \mathbf{G} -bundle over $\mathbb{A}_R^1 := \text{Spec } R[t]$ such that \mathcal{F} is trivial over the complement of a closed subscheme finite over $\text{Spec } R$. Then \mathcal{F} is trivial.*

This theorem is similar to [PSV, Thm. 1.3] and to [FP, Thm. 3]. It will be proved in Section 6.

Remark 2.2. Note that we do not require the ring R to be regular. Thus one has to be careful with the definition of split group scheme: one should additionally require that the root spaces of the split maximal torus are free R -modules (see [DG, Exp. XXII, Def. 1.13]).

2.1. Example: quadratic forms. Let X and x be as in Theorem 1. Denote the local ring $\mathcal{O}_{X,x}$ by R . Define the split quadratic form over R as follows

$$Q_n = x_1x_{m+1} + \dots + x_mx_{2m} \text{ if } n = 2m$$

and

$$Q_n = x_1x_{m+1} + \dots + x_mx_{2m} + x_{2m+1}^2 \text{ if } n = 2m+1.$$

Recall (see e.g. [Knu, Ch. 4, Sect. 3]) that if n is odd and Q is a quadratic form with coefficients in R , then one can define its half-discriminant (which is just $1/2$ times the discriminant if 2 is invertible in R). The following is a corollary of Theorem 1.

Theorem 3. *Let $Q = \sum_{i,j} q_{ij}x_i x_j$ be a quadratic form in n variables with coefficients in R such that its discriminant is invertible in R if n is even, and its half-discriminant is invertible in R if n is odd. Assume that there is a linear transformation with coefficients in the fraction field of R , taking Q to Q_n . Then there is a linear transformation with coefficients in R taking Q to Q_n .*

Proof. Let \mathbf{O}_n be the R -group scheme of orthogonal transformations of Q_n . The scheme of isomorphisms $\text{Isom}(Q, Q_n)$ is a principal \mathbf{O}_n -bundle over $\text{Spec } R$. This bundle is locally trivial in the fppf topology. (Note that if n is odd and $2 \notin R^\times$, then \mathbf{O}_n is not smooth over R .) Thus, we only need to show that the natural morphism $H_{\text{fppf}}^1(R, \mathbf{O}_n) \rightarrow H_{\text{fppf}}^1(K, \mathbf{O}_n)$ has a trivial kernel.

Let \mathbf{SO}_n be the special orthogonal group scheme associated to Q_n (see [Knu, Ch. IV, Sect. 5] for the correct definition in case when 2 is not invertible in R).

Then \mathbf{SO}_n is a split reductive group scheme. If n is odd, we have $\mathbf{O}_n \simeq \mu_2 \times \mathbf{SO}_n$, where μ_2 is the group scheme of square roots of unity. The natural morphism $H_{\text{fppf}}^1(R, \mathbf{SO}_n) \rightarrow H_{\text{fppf}}^1(K, \mathbf{SO}_n)$ has a trivial kernel by Theorem 1 (recall that for smooth group schemes there is no difference between fppf principal bundles and étale principal bundles). On the other hand, we have $H_{\text{fppf}}^1(R, \mu_2) = (R^\times)^2/R^\times$ (because $H^1(R, \mathcal{O}_R^\times) = 1$, since R is factorial). Similarly, $H_{\text{fppf}}^1(K, \mu_2) = (K^\times)^2/K^\times$. It follows now from factoriality of R that the morphism $H_{\text{fppf}}^1(R, \mu_2) \rightarrow H_{\text{fppf}}^1(K, \mu_2)$ has a trivial kernel. This completes the proof in the case, when n is odd.

If n is even, we have an exact sequence $1 \rightarrow \mathbf{SO}_n \rightarrow \mathbf{O}_n \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1$ by [Knu, Ch. 4, Prop. 5.2.2]. This gives an exact sequence of cohomology

$$\begin{array}{ccccccc} \mathbf{O}_n(R) & \longrightarrow & \mathbb{Z}/2\mathbb{Z}(R) & \longrightarrow & H_{\text{fppf}}^1(R, \mathbf{SO}_n) & \longrightarrow & H_{\text{fppf}}^1(R, \mathbf{O}_n) \longrightarrow H_{\text{fppf}}^1(R, \mathbb{Z}/2\mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbf{O}_n(K) & \longrightarrow & \mathbb{Z}/2\mathbb{Z}(K) & \longrightarrow & H_{\text{fppf}}^1(K, \mathbf{SO}_n) & \longrightarrow & H_{\text{fppf}}^1(K, \mathbf{O}_n) \longrightarrow H_{\text{fppf}}^1(K, \mathbb{Z}/2\mathbb{Z}). \end{array}$$

Note that the two right vertical arrows are injective. Next, the morphism $\mathbf{O}_n(K) \rightarrow \mathbb{Z}/2\mathbb{Z}(K)$ is surjective (again by [Knu, Ch. 4, Prop. 5.2.2]). By Theorem 1 the middle vertical arrow has a trivial kernel. Now an easy diagram chase proves the claim. \square

3. OUTLINE OF THE PROOF OF THEOREM 1

In this section we introduce the main ideas of the proof and reduce the theorem to a sequence of propositions to be proved in subsequent sections. Let X and x be as in Section 2. Set $U := \text{Spec } \mathcal{O}_{X,x}$. We may identify the unique closed point of U with x ; denote the residue field of x by $k(x)$.

Conventions. Let T be a scheme of pure dimension. When we say “ $D \subset T$ is a divisor”, we mean that D is an effective Cartier divisor on T . In other words, $D \subset T$ is a closed subscheme (not necessarily reduced), which is Zariski locally over T given by sections of \mathcal{O}_T . We never consider non-effective divisors in this paper.

Let us give a very brief overview of the proof first. The first step in the proof is to fiber a neighborhood of x in X into curves. Thus we will choose an appropriate neighborhood X' of x in X and a smooth fibration $X' \rightarrow S$ of relative dimension one. We will extend \mathcal{G} to a \mathbf{G} -bundle \mathcal{F} over X' such that \mathcal{G} is trivial over the complement of a subscheme finite over S . Next, we will pull \mathcal{F} back to an open subset of $X' \times_S U$. Then, we descend the bundle obtained to \mathbb{A}_U^1 , reducing Theorem 1 to Theorem 2.

Only the first step is significantly different from the equal characteristic case. In particular, we use the fact that a generically trivial principal bundle can be reduced to a Borel subgroup on the complement of a codimension two subscheme, see Lemma 4.9.

3.1. Quasi-elementary fibrations. The notion of an elementary fibration was introduced in [SGA, Exp. XI, Def. 3.1]. The following notion is a weak version of elementary fibration: we only assume that the projection is smooth over the open part, we do not require the fibers to be integral, and we only require the divisor to be finite surjective over the base (see also [PSV, Def. 2.1]).

Definition 3.1. A *quasi-elementary fibration* is a morphism of schemes $p : X' \rightarrow S$ that can be included in a commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{j} & \bar{X} & \xleftarrow{i} & Y \\ & \searrow p & \downarrow \bar{p} & \swarrow q & \\ & & S & & \end{array}$$

satisfying the following conditions

- (1) p is affine and smooth;
- (2) \bar{X} is a regular scheme of pure dimension;
- (3) \bar{p} is flat projective of pure relative dimension one;
- (4) j is an open embedding, i is a closed embedding, and $X' = \bar{X} - Y$;
- (5) q is finite surjective;
- (6) Y is a divisor in \bar{X} .

Proposition 3.2. Let \mathcal{G} be a generically trivial \mathbf{G} -bundle over $U = \text{Spec } \mathcal{O}_{X,x}$. Then there are

- an open affine subscheme $X' \subset X$ containing x ;
- a quasi-elementary fibration $p : X' \rightarrow S$ with S connected and smooth over B ;
- a principal divisor $Z' \subset X'$ finite over S ;
- a \mathbf{G} -bundle \mathcal{F} over X' extending \mathcal{G} and such that \mathcal{F} is trivial over $X' - Z'$;
- a finite surjective S -morphism $X' \rightarrow \mathbb{A}_S^1$.

The proof of this proposition is given in Section 4. Fix the data provided by the proposition.

3.2. Nice triples. Recall the notion of a nice triple from [PSV, Def. 3.1].

Definition 3.3. A *nice triple* over U is a triple $(q_U : \mathcal{X} \rightarrow U, f, \Delta)$, where \mathcal{X} is an irreducible scheme smooth over U and such that all its fibers are of pure dimension one, $f \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is such that its zero locus \mathcal{Z} is finite over U , and $\Delta : U \rightarrow \mathcal{X}$ is a section of q_U such that $\Delta^*(f) \neq 0$. These data is subject to the condition that there exists a finite U -morphism $\mathcal{X} \rightarrow \mathbb{A}_U^1$.

Remark 3.4. The condition that there exists a finite U -morphism $\mathcal{X} \rightarrow \mathbb{A}_U^1$ shows that \mathcal{X} is affine. Thus finiteness of \mathcal{Z} is equivalent to the condition that

$$\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/f \cdot \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$$

is finite as a $\Gamma(U, \mathcal{O}_U)$ -module.

Proposition 3.5. Let \mathcal{G} be a \mathbf{G} -bundle over U trivial over the generic point of U . Then there is a nice triple $(q_U : \mathcal{X} \rightarrow U, f, \Delta)$ and a \mathbf{G} -bundle \mathcal{E} over \mathcal{X} such that

- (1) $\Delta^*\mathcal{E} \simeq \mathcal{G}$;
- (2) \mathcal{E} is trivial over the complement of the zero locus \mathcal{Z} of f .

Moreover, if the field $k(x)$ is finite, we may choose this nice triple so that

- (3) There is at most one point $z \in \mathcal{Z}_x$ rational over $k(x)$;
- (4) For any integer $r \geq 1$ one has

$$\#\{z \in \mathcal{Z}_x \mid \deg[z : x] = r\} \leq \#\{z \in \mathbb{A}_x^1 \mid \deg[z : x] = r\},$$

where $\#A$ denotes the number of elements of the finite set A .

This proposition is derived from Proposition 3.2. The proof is similar to the considerations of Theorem 3.3 and Section 6 of [Pan1], see also [PSV]. For the reader's benefit, we give a proof in Section 5.

Let (q_U, f, Δ) be a nice triple provided by the above proposition. We may assume that f vanishes at $\Delta(x)$, otherwise the statement of Theorem 1 is obvious. If $k(x)$ is finite, then by condition (3) of the above proposition $\Delta(x)$ is the only $k(x)$ -rational point of \mathcal{Z}_x . Set $R := \mathcal{O}_{X,x}$ so that $U = \text{Spec } R$.

Proposition 3.6. *Let (q_U, f, Δ) be a nice triple over U such that $\Delta(x) \in \mathcal{Z}$. Assume that this nice triple satisfies conditions (3) and (4) of the above proposition if $k(x)$ is finite. Then there are a finite surjective U -morphism $\sigma : \mathcal{X} \rightarrow \mathbb{A}_U^1$, a monic polynomial $h \in R[t]$ vanishing on $\sigma(\mathcal{Z})$, and an element $g \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ such that*

- (1) *the morphism $\sigma_g := \sigma|_{\mathcal{X}_g}$ is étale;*
- (2) *the data $(R[t], \sigma_g^* : R[t] \rightarrow \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g, h)$ satisfies the hypothesis of [CTO, Prop. 2.6], that is, $R[t]$ is Noetherian, $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g$ is finitely generated as an $R[t]$ -algebra, $\sigma_g^*(h)$ is not a zero divisor in $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g$, and σ_g^* induces an isomorphism*

$$R[t]/(h) \simeq \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g / \sigma_g^*(h) \cdot \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g;$$

- (3) $\Delta(U) \cup \mathcal{Z} \subset \mathcal{X}_g$.

Proof. If $k(x)$ is finite and R contains a field, then this is Theorem 3.4 of [Pan1]. However, the fact that R contains a field is not used in the proof as one easily checks. Similarly, in the case of infinite field $k(x)$ this is Theorem 3.4 of [PSV]. Again, one checks that the requirement that R contains a field is not used in the proof. \square

3.3. End of the proof of Theorem 1.

Proposition 3.7. *Let \mathcal{G} be a generically trivial \mathbf{G} -bundle over $U = \text{Spec } \mathcal{O}_{X,x}$. Then there is a \mathbf{G} -bundle \mathcal{F} over \mathbb{A}_U^1 such that*

- \mathcal{F} is trivial over the complement of a closed subscheme $\mathcal{Y} \subset \mathbb{A}_U^1$ such that \mathcal{Y} is finite over U ;
- $\mathcal{F}|_{0 \times U} \simeq \mathcal{G}$.

Proof. Let $(q_U : \mathcal{X} \rightarrow U, f, \Delta)$ and \mathcal{E} be from Proposition 3.5. Let σ , h , and g be from Proposition 3.6. After performing an affine transformation of \mathbb{A}_U^1 , we may assume that $\Delta^*(\sigma)$ coincides with the closed embedding $0 \times U \hookrightarrow \mathbb{A}_U^1$. Condition (2) of Proposition 3.6 together with [CTO, Prop. 2.6] shows that the diagram

$$\begin{array}{ccc} \mathcal{X}_{g\sigma^*(h)} & \longrightarrow & \mathcal{X}_g \\ \downarrow & & \sigma_g \downarrow \\ (\mathbb{A}_U^1)_h & \longrightarrow & \mathbb{A}_U^1 \end{array}$$

can be used to glue principal \mathbf{G} -bundles in the following sense: given a \mathbf{G} -bundle over $(\mathbb{A}_U^1)_h$, a \mathbf{G} -bundle over \mathcal{X}_g , and an isomorphism of their pullbacks to $\mathcal{X}_{g\sigma^*(h)}$, we can glue the bundles to make a \mathbf{G} -bundle over \mathbb{A}_U^1 . In particular, since $\mathcal{X}_{g\sigma^*(h)} \subset \mathcal{X}_f$, we can glue $\mathcal{E}|_{\mathcal{X}_g}$ with the trivial \mathbf{G} -bundle over $(\mathbb{A}_U^1)_h$ to make the desired \mathbf{G} -bundle \mathcal{F} over \mathbb{A}_U^1 .

Clearly, all the conditions of the proposition are satisfied with $\mathcal{Y} := \{h = 0\}$. \square

Now Theorem 2 completes the proof of Theorem 1.

4. QUASI-ELEMENTARY FIBRATIONS: PROOF OF PROPOSITION 3.2

In this section we will prove Proposition 3.2 but we need some generalities first. Through the end of the section, all schemes are assumed to be Noetherian.

Convention. Let S be a scheme, let T_i be S -schemes, and let $s \in S$ be a point. By $\text{shrinking } (S, s)$ we mean replacing S by a Zariski neighborhood S' of s and replacing each T_i by $T_i \times_S S'$.

Lemma 4.1. *Let $\varphi : T \rightarrow S$ be a projective morphism with fibers of dimension one (but not necessarily of pure dimension), let $s \in S$ be a closed point. Let $T_1, T_2 \subset T$ be closed subschemes finite over S and such that $T_1 \cap T_2 = \emptyset$. Then*

- (a) *If \mathcal{L} is an ample line bundle over T/S , then for all large N we may shrink (S, s) so that we can find $\sigma \in H^0(T, \mathcal{L}^{\otimes N})$ such that σ vanishes on T_1 and does not vanish at any point of T_2 .*
- (b) *After shrinking (S, s) , we can find a divisor $D \subset T$ finite over S such that $T_1 \subset D$, $T_2 \cap D = \emptyset$. Moreover, we may assume that $T - D$ is affine over S .*
- (c) *After shrinking (S, s) , we can find a finite surjective S -morphism $\Pi : T \rightarrow \mathbb{P}_S^1$ such that $\Pi(T_1) \subset 0 \times S$, $\Pi(T_2) \subset \infty \times S$.*

Proof. For part (a), consider $T_0 := T_1 \cup (T_2)_s$ and notice that $R^1\varphi_*(\mathcal{L}^{\otimes N}(-T_0))$ vanishes in a neighborhood of s for large N . Thus, after shrinking (S, s) , we can find a section of $\mathcal{L}^{\otimes N}$ such that this section vanishes on T_1 and does not vanish at any point of $(T_2)_s$. It remains to shrink (S, s) again.

For part (b), choose an ample line bundle \mathcal{L} over T/S . Enlarging T_2 , we may assume that it contains a closed point in each dimension one irreducible component of T_s . Let σ be a section of $\mathcal{L}^{\otimes N}$ provided by the first part, let D be its divisor of zeroes. Then the fiber of D over s is finite. Since D is projective over S , the dimensions of fibers are semicontinuous. Thus, after shrinking (S, s) , we may assume that D is quasi-finite over S . Since D is projective over S , it is finite over S . To get $T - D$ affine over S , we just need to start with a very ample \mathcal{L} .

For part (c), we may assume that each of T_1 and T_2 contains at least one point on each irreducible dimension one component of T_s . Let \mathcal{L} be a very ample line bundle on T/S . Thus, by part (a), shrinking (S, s) and replacing \mathcal{L} by its power, we can find a section τ_1 of \mathcal{L} such that τ_1 vanishes on T_1 but not at the points of T_2 . Let T' be the zero set of τ_1 .

As in part (b), we may assume that T' is finite over S . Shrinking (S, s) and applying part (a) again, we see that there is a section τ_2 of $\mathcal{L}^{\otimes N}$ for some $N > 0$ such that τ_2 vanishes on T_2 but not at the points of T' .

Consider the projective morphism $\Pi : T \rightarrow \mathbb{P}_S^1$ given by $[\tau_2 : \tau_1^N]$. Its restriction to T_s is finite because it is a morphism of one-dimensional projective schemes $T_s \rightarrow \mathbb{P}_s^1$ such that both the preimage of zero and the preimage of infinity intersect all one-dimensional components of T_s . Thus, shrinking (S, s) , we may assume that Π is finite. Clearly, we have $\Pi(T_1) \subset 0 \times S$ and $\Pi(T_2) \subset \infty \times S$.

It remains to show that Π is surjective. Since Π is closed (being finite), we only need to check that for any generic point ω of S the base-changed morphism $\Pi_\omega : T_\omega \rightarrow \mathbb{P}_\omega^1$ is surjective. If not, then its image is finite, so Π_ω cannot be finite because T_ω is one-dimensional. This contradiction completes the proof of surjectivity. \square

Let us define the dimension of the empty scheme to be -1 . The following proposition follows easily from results of [Po].

Proposition 4.2. *Assume that T_1, \dots, T_n are non-empty locally closed subschemes of \mathbb{P}_k^N , where k is a field. Let $T' \subset T$ be smooth locally closed subschemes of \mathbb{P}_k^N , and let F be a finite set of closed points of \mathbb{P}_k^N . Assume that for all i we have $T_i \not\subset F$. Then there is a hypersurface $H \subset \mathbb{P}_k^N$ such that the scheme theoretic intersections $H \cap T$ and $H \cap T'$ are smooth, $F \subset H$, and for $i = 1, \dots, n$ we have $\dim(H \cap T_i) < \dim T_i$.*

Proof. Assume first that k is a finite field. Replacing each T_i by the set of its irreducible components, we may assume that each T_i is irreducible. For $i = 1, \dots, n$ choose a closed point $p_i \in T_i - F$. Now we apply [Po, Thm. 1.3] (see also the proof of [Po, Thm. 3.3]), to find a hypersurface H such that

- $H \cap T$ and $H \cap T'$ are smooth;
- $F \subset H$;
- For all i we have $p_i \notin H$.

Clearly, this hypersurface H satisfies conditions of the proposition.

If k is an infinite field, the statement follows easily from Bertini's Theorem. \square

4.1. Weighted blow-ups. Denote by $\mathbb{P}_{\mathbb{Z}}(l_0, \dots, l_m)$ the weighted projective space, that is,

$$\mathbb{P}_{\mathbb{Z}}(l_0, \dots, l_m) := \text{Proj}(\mathbb{Z}[x_0, \dots, x_m]), \quad \deg x_i = l_i.$$

For a scheme S , set $\mathbb{P}_S(l_0, \dots, l_m) := \mathbb{P}_{\mathbb{Z}}(l_0, \dots, l_m) \times S$.

Let Z be a reduced scheme, let \mathcal{L} be an invertible sheaf on Z and let

$$\sigma_i \in H^0(Z, \mathcal{L}^{\otimes l_i}), \quad i = 0, \dots, m.$$

Let Z_0 be the intersection of the zero loci of σ_i . The sections σ_i give rise to a morphism

$$Z - Z_0 \xrightarrow{\mu} \mathbb{P}_{\mathbb{Z}}(l_0, \dots, l_m).$$

Denote by $\text{Bl}_{\sigma_0, \dots, \sigma_m}(Z)$ the closure of the graph of μ in $\mathbb{P}_Z(l_0, \dots, l_m)$. We view it as a scheme with reduced scheme structure. We have a projection

$$\lambda : \text{Bl}_{\sigma_0, \dots, \sigma_m}(Z) \rightarrow Z;$$

this is a projective morphism. Note the following easy lemma.

Lemma 4.3. *The base change of λ with respect to the inclusion $Z - Z_0 \hookrightarrow Z$ is an isomorphism.*

We will consider weighted blow-ups only in the case, when $l_0 = 1$. For a scheme S , denote by \hat{A}_S the open subset of $\mathbb{P}_S(1, l_1, \dots, l_m)$ given by $x_0 \neq 0$. If $S = \text{Spec } A$, then $\hat{A}_S = \text{Spec } A[y_1, \dots, y_m]$, where $y_i = x_i/x_0^{l_i}$. Thus, for any S , we have a canonical isomorphism $\hat{A}_S = \mathbb{A}_S^m$, in particular, \hat{A}_S is smooth over S .

The following lemma says, essentially, that the blow-up of a divisor is isomorphic to this divisor.

Lemma 4.4. *Assume that Z is separated and that Z_0 is of pure codimension $m + 1$ in Z . Let Z_1 be the intersection of the zero loci of $\sigma_1, \dots, \sigma_m$. Assume that Z_1 is reduced. Set $Z'_1 := Z_1 \times (1 : 0 : \dots : 0) \subset \mathbb{P}_Z(1, l_1, \dots, l_m)$. Then $Z'_1 \subset \text{Bl}_{\sigma_0, \dots, \sigma_m}(Z)$ and λ induces an isomorphism $Z'_1 \simeq Z_1$.*

Proof. Put $Z'_0 := Z_0 \times (1 : 0 : \dots : 0) \subset \mathbb{P}_Z(1, l_1, \dots, l_m)$. It is clear that $\lambda^{-1}(Z_1 - Z_0) = Z'_1 - Z'_0$. Thus $Z'_1 - Z'_0 \subset \text{Bl}_{\sigma_0, \dots, \sigma_m}(Z)$. By our assumption on codimensions, $Z'_1 - Z'_0$ is dense in Z'_1 . It follows that $Z'_1 \subset \text{Bl}_{\sigma_0, \dots, \sigma_m}(Z)$. It remains to show that the restriction $\lambda|_{Z'_1}$ is the standard isomorphism $Z'_1 \rightarrow Z_1$. This is certainly true over $Z'_1 - Z'_0$, so our statement follows from separatedness of Z'_1 . \square

The following lemma is saying that the blow-up of a regular scheme is regular, at least over $\hat{\mathbb{A}}_S$ (cf. [Gro1, Prop. 19.4.8]).

Lemma 4.5. *Let Z and Z_0 be regular schemes. Assume that Z is an excellent scheme. Then $\mathrm{Bl}_{\sigma_0, \dots, \sigma_m}(Z) \cap \hat{\mathbb{A}}_Z$ is a regular scheme.*

Proof. The statement is local over Z , so we may assume that $Z = \mathrm{Spec} A$ is affine and that \mathcal{L} is a trivial line bundle. Choosing a trivialization of \mathcal{L} , we may view σ_i as elements of A . Since Z is excellent, $Z' := \mathrm{Bl}_{\sigma_0, \dots, \sigma_m}(Z) \cap \hat{\mathbb{A}}_Z$ is also so. Thus we only need to show that Z' is regular at each closed point x . Let $x \in Z'$ lie over $z \in Z = \mathrm{Spec} A$. Viewing z as a maximal ideal of A , we may assume that $z \supset (\sigma_0, \dots, \sigma_m)$ (otherwise we are done by Lemma 4.3). Let $d\sigma_i$ be the image of σ_i in the cotangent space T_z^* of Z at z . Since Z_0 is regular, we see that $d\sigma_0, \dots, d\sigma_m$ are linearly independent.

Let T_x^* be the cotangent space of $\hat{\mathbb{A}}_Z$ at x . We have $T_x^* = (T_z^* \otimes_{k(z)} k(x)) \oplus V$, where V is the $k(x)$ -vector space with basis dy_1, \dots, dy_m . Let T^* be the cotangent space of Z' at x . We have the surjective projection $T_x^* \rightarrow T^*$; denote the kernel of this projection by K . Since $\sigma_i - y_1\sigma_0^{i_1}$ vanishes on Z' , and $d\sigma_i$ are linearly independent, the dimension of K is at least m . Thus the dimension of T^* is less or equal to the dimension of Z . Since the dimension of Z' is equal to the dimension of Z , we see that Z' is regular at x . \square

4.2. Constructing quasi-elementary fibrations. Let $X \rightarrow \mathrm{Spec} B$ and $x \in X$ be as in Section 2. That is, B is an excellent DVR, $b \in \mathrm{Spec} B$ is the closed point. Also, X is integral, flat and projective over $\mathrm{Spec} B$, and satisfies conditions (I) and (II) of Section 2; the projection $X \rightarrow \mathrm{Spec} B$ is smooth at x . In this section we prove the following proposition.

Proposition 4.6. *Let X^0 be an open subscheme of X such that $x \in X^0$. Assume also that the intersection of X^0 with the fiber X_b is dense in this fiber. Assume that Z is a closed subset of X^0 of codimension at least two. Then there is an open subscheme $X' \subset X^0$ containing x , a connected B -scheme S smooth over B , and an S -morphism $p : X' \rightarrow S$ such that p is a quasi-elementary fibration and $Z \cap X'$ is finite over S .*

Proof. The proof is somewhat technical but it follows the same strategy as the proofs of [Pan1, Prop. 2.3], [PSV, Prop. 2.1] and of Artin's result [SGA, Exp. XI, Prop. 3.3].

Step 1. We may assume that X^0 is smooth over $\mathrm{Spec} B$ (use condition (I) and openness of smoothness). Set $Y^0 := X - X^0$. Set $n = \dim X - 1 = \dim X_b$. Note that $\dim Y_b^0 \leq n - 1$. Denote by \bar{Z} the Zariski closure of Z in X . Then $(\bar{Z})_b$ is the intersection of \bar{Z} with X_b , which is in general larger, than the closure of Z_b . In any case,

$$\dim(\bar{Z})_b \leq \dim \bar{Z} \leq n - 1.$$

Lemma 4.7. *There is a B -embedding $X \hookrightarrow \mathbb{P}_B^N$, a hyperplane $H_0 \subset \mathbb{P}_{k(b)}^N$, and hypersurfaces $H_1, \dots, H_{n-1} \subset \mathbb{P}_{k(b)}^N$, satisfying the following conditions (we put $L := H_1 \cap \dots \cap H_{n-1}$).*

- $x \notin H_0$, $x \in L$;
- $(X^{\mathrm{sing}})_b \cap L = \emptyset$;
- $X_b^0 \cap L$ is smooth of dimension one over b ;
- $Y_b^0 \cap L$ is finite;
- $(\bar{Z})_b \cap L$ is finite;

- $(\bar{Z} - Z)_b \cap L = \emptyset$;
- $(\bar{Z})_b \cap L \cap H_0 = \emptyset$;
- $X_b \cap L \cap H_0$ is finite and étale over b .

Proof. Choose any B -embedding $X \hookrightarrow \mathbb{P}_B^N$. By Proposition 4.2 there is a hypersurface $H_0 \subset \mathbb{P}_{k(b)}^N$ such that (i) $x \notin H_0$; (ii) $\dim(Y_b^0 \cap H_0) \leq n-2$, $\dim(\bar{Z})_b \cap H_0 \leq n-2$, and $X_b^0 \cap H_0$ is smooth. Replacing the embedding $X \hookrightarrow \mathbb{P}_B^N$ by its composition with the $(\deg H_0)$ -fold Veronese embedding, we may assume that H_0 is a hyperplane.

Next, we repeatedly use Proposition 4.2 to find hypersurfaces H_1, \dots, H_{n-1} . We use the following facts:

- $\dim(X^{sing})_b \leq n-2$ by condition (II) on X ;
- $x \notin X^{sing}$ by assumption;
- $\dim(\bar{Z} - Z)_b \leq \dim(\bar{Z} - Z) \leq n-2$;
- $x \notin \bar{Z} - Z$ because Z is closed in X_0 and $x \in X_0$.

To achieve that $X_b \cap L \cap H_0$ is finite and étale over b it is enough to require that $X_b^0 \cap L \cap H_0$ is finite and smooth over b and that $Y_b^0 \cap L \cap H_0 = \emptyset$. \square

Step 2. Let $\sigma'_i \in H^0(\mathbb{P}_{k(b)}^N, \mathcal{O}(l_i))$ be an equation of H_i , where $l_i := \deg H_i$. We can extend each σ'_i to $\tilde{\sigma}_i \in H^0(\mathbb{P}_B^N, \mathcal{O}(l_i))$. Set $\mathcal{L} := \mathcal{O}_{\mathbb{P}_B^N}(1)|_X$, $\sigma_i := \tilde{\sigma}_i|_X$, so that $\sigma_i \in H^0(X, \mathcal{L}^{\otimes l_i})$. Set $\bar{X} := \text{Bl}_{\sigma_0, \dots, \sigma_{n-1}}(X)$ (see Section 4.1).

Denote the zero locus of $\tilde{\sigma}_i$ by \tilde{H}_i and set $\tilde{L} := \tilde{H}_1 \cap \dots \cap \tilde{H}_{n-1}$.

Let $\lambda : \bar{X} \rightarrow X$ be the canonical morphism. Denote by E the exceptional locus of λ , that is, $E = \lambda^{-1}(X \cap \tilde{L} \cap \tilde{H}_0)$. By Lemma 4.3, λ induces an isomorphism $\bar{X} - E = X - X \cap \tilde{L} \cap \tilde{H}_0$. Set $\hat{Z} := \lambda^{-1}(\bar{Z})$, $\hat{Y} := \lambda^{-1}(Y^0)$. We identify x with its unique λ -preimage in \bar{X} , see Lemma 4.3.

In the notation of Section 4.1, we have a projective morphism $\bar{p} : \bar{X} \rightarrow S := \mathbb{P}_B(1, l_1, \dots, l_{n-1})$. Consider the point $(1:0:\dots:0) \in \mathbb{P}_{k(b)}(1, l_1, \dots, l_{n-1})$; denote its image in S by 1_b . We have $\bar{p}(x) = 1_b$. Set $F := \bar{p}^{-1}(1_b)$.

Step 3. We claim that

- (1) λ induces an isomorphism $F \simeq X_b \cap L$;
- (2) \bar{X} is regular at the points of F ;
- (3) \bar{p} is flat at the points of F ;
- (4) $\bar{X}^s \cap F$ is finite, where \bar{X}^s is the set, where \bar{p} is not smooth;
- (5) $E \cap F$, $\hat{Y} \cap F$, and $\hat{Z} \cap F$ are finite;
- (6) $\hat{Z} \cap F = \lambda^{-1}(Z) \cap F$;
- (7) $\hat{Z} \cap E \cap F = \hat{Z} \cap \hat{Y} \cap F = \hat{Z} \cap \bar{X}^s \cap F = \emptyset$.

Indeed, first of all $X \cap \tilde{L}$ and $X \cap \tilde{L} \cap \tilde{H}_0$ are complete intersections in X . In particular, they are flat over $\text{Spec } B$ by [Mat, Thm. 23.1]. We see that $X \cap \tilde{L}$ is reduced (being generically reduced) and that $X \cap \tilde{L} \cap \tilde{H}_0$ is smooth over $\text{Spec } B$. Now (2) follows from Lemma 4.5. Further, Lemma 4.4 claims that λ induces an isomorphism

$$X \cap L \simeq \bar{p}^{-1}(\text{Spec } B \times (1 : 0 : \dots : 0))$$

and (1) follows by taking the fiber over b .

Next, (3) follows from [Mat, Thm. 23.1]. To prove (4) note that \bar{p} , being flat, is smooth exactly where the fiber is smooth. Now use (1) and Lemma 4.7. The remaining statements follow easily from (1) and the respective properties of L and H_0 .

Step 4. After shrinking $(S, 1_b)$ and replacing \bar{X} , E , \hat{Y} , and \hat{Z} by their intersections with $\bar{p}^{-1}(S)$, we may assume that

- (1) S is connected and smooth over B ;
- (2) \bar{X} is regular;
- (3) \bar{p} is flat of pure relative dimension one;
- (4) \bar{X}^s , E , \hat{Y} , and \hat{Z} are finite over S ;
- (5) There is a divisor $Y \subset \bar{X}$ finite over S such that $Y \supset E \cup \bar{X}^s \cup \hat{Y}$, $Y \cap \hat{Z} = \emptyset$, $x \notin Y$, and $\bar{X} - Y$ is affine.

Indeed, (1) is obvious, (2) follows from the fact that the set of points, where \bar{X} is regular is open in \bar{X} (because B is excellent) and the fact that \bar{p} is closed. Next, flatness in (3) follows from the openness of flatness. Now the fact that \bar{p} is of pure relative dimension one follows because \bar{X} is equidimensional of dimension $n + 1$, while S is of dimension n . Next, (4) follows because the dimensions of fibers of a projective morphism are semicontinuous and a quasi-finite projective morphism is finite; finally, (5) follows from Lemma 4.1(b).

Step 5. The restriction of λ to $X' := \bar{X} - Y$ is an open embedding, so we can identify X' with an open subset of X^0 . It is clear that $\bar{p}|_{X'} : X' \rightarrow S$ is a quasi-elementary fibration.

Also, shrinking $(S, 1_b)$ again if necessary, we may assume that under the identification of X' and $\lambda(X')$ we have $\hat{Z} = Z \cap X'$, so $Z \cap X'$ is finite over S . \square

4.3. Proof of Proposition 3.2.

Lemma 4.8. *We can find a regular open subscheme $X^0 \subset X$ such that $x \in X^0$, X^0 intersects all irreducible components of X_b , and \mathcal{G} can be extended to a \mathbf{G} -bundle \mathcal{G}^0 over X^0 .*

Proof. We can find $X_1 \subset X$ such that X_1 is open in X , $x \in X_1$, and \mathcal{G} can be extended to a \mathbf{G} -bundle \mathcal{G}_1 over X_1 . Since \mathcal{G} is generically trivial, \mathcal{G}_1 is trivial on the complement of a proper closed subscheme $Z_1 \subset X_1$. Since X_b is smooth at x , we see that x lies on a single irreducible component of X_b . Thus we may assume that X_1 does not intersect irreducible components of X_b other than that containing x .

Let \bar{Z}_1 be the Zariski closure of Z_1 in X . We have

$$\dim(\bar{Z}_1 - Z_1) < \dim Z \leq n.$$

It follows that $\bar{Z}_1 - Z_1$ cannot contain an irreducible component of X_b . Thus \bar{Z}_1 cannot contain irreducible components of X_b other than the component containing x . Consider the trivial \mathbf{G} -bundle over $X - \bar{Z}_1$. We can glue it with \mathcal{G}_1 to make a \mathbf{G} -bundle \mathcal{G}_2 over $X_2 := (X - \bar{Z}_1) \cup X_1$. One now takes X^0 to be the regular locus of X_2 and sets $\mathcal{G}^0 := \mathcal{G}_2|_{X^0}$. It follows from the construction and property (II) of $\pi : X \rightarrow \text{Spec } B$, that X^0 satisfies the requirements of the lemma. \square

Fix such X^0 and \mathcal{G}^0 . Since \mathbf{G} is split, there is a split maximal torus $\mathbf{T} \subset \mathbf{G}$ and a Borel subgroup $\mathbf{B} \subset \mathbf{G}$ containing \mathbf{T} . Fix such $\mathbf{T} \subset \mathbf{B}$. Recall that a \mathbf{B} -bundle \mathcal{B} over a B -scheme T induces a \mathbf{G} -bundle $\mathbf{G} \times^{\mathbf{B}} \mathcal{B}$. We say that a \mathbf{G} -bundle \mathcal{G} over T can be *reduced* to T , if there is a \mathbf{B} -bundle \mathcal{B} over T such that $\mathbf{G} \times^{\mathbf{B}} \mathcal{B}$ is isomorphic to \mathcal{G} as a \mathbf{G} -bundle.

Lemma 4.9. *\mathcal{G}^0 can be reduced to \mathbf{B} over $X^0 - Z$, where Z is closed and of codimension at least two in X^0 .*

Proof. The idea of the proof is very simple: the \mathbf{B} -reductions of \mathcal{G}^0 are given by sections of $\mathbf{B} \backslash \mathcal{G}^0$, so if we knew that this space is projective over X^0 , the statement

would be obvious. Since we only know projectivity after étale base changed, we will have to employ étale descent.

The quotient $\mathbf{B} \setminus \mathbf{G}$ exists and is projective over $\mathrm{Spec} B$. Moreover, $\mathbf{G} \rightarrow \mathbf{B} \setminus \mathbf{G}$ is a principal \mathbf{B} -bundle and the induced \mathbf{G} -bundle $\mathbf{G} \times^{\mathbf{B}} \mathbf{G}$ is canonically trivialized.

Let $\varphi : \tilde{X} \rightarrow X^0$ be a surjective étale morphism such that $\tilde{\mathcal{G}} := \mathcal{G}^0 \times_{X^0} \tilde{X}$ is a trivial \mathbf{G} -bundle. Choose a trivialization of $\tilde{\mathcal{G}}$. It follows that the quotient $\mathbf{B} \setminus \tilde{\mathcal{G}} \approx (\mathbf{B} \setminus \mathbf{G}) \times_{\mathrm{Spec} B} \tilde{X}$ exists and is projective over \tilde{X} . Moreover, $\tilde{\mathcal{G}} \rightarrow \mathbf{B} \setminus \tilde{\mathcal{G}}$ is a \mathbf{B} -bundle, such that we have isomorphisms of \mathbf{G} -bundles over $\tilde{\mathcal{G}}$

$$(1) \quad \mathbf{G} \times^{\mathbf{B}} \tilde{\mathcal{G}} = \mathbf{G} \times (\mathbf{B} \setminus \tilde{\mathcal{G}}) = \tilde{\mathcal{G}} \times_{\tilde{X}} (\mathbf{B} \setminus \tilde{\mathcal{G}}) = \mathcal{G}^0 \times_{X^0} (\mathbf{B} \setminus \tilde{\mathcal{G}}).$$

In other words, $\tilde{\mathcal{G}} \rightarrow \mathbf{B} \setminus \tilde{\mathcal{G}}$ is a \mathbf{B} -reduction of $\mathcal{G}^0 \times_{X^0} (\mathbf{B} \setminus \tilde{\mathcal{G}}) \rightarrow \mathbf{B} \setminus \tilde{\mathcal{G}}$.

Since \mathcal{G}^0 is generically trivial, we can find a dense open subset $U^0 \subset X^0$ and a section $s : U^0 \rightarrow \mathcal{G}^0$ of \mathcal{G}^0 over U^0 . Put $\tilde{U} := U^0 \times_{X^0} \tilde{X}$ and let s' be the composition

$$\tilde{U} \xrightarrow{s \times \mathrm{Id}_{\tilde{X}}} \tilde{\mathcal{G}} \rightarrow \mathbf{B} \setminus \tilde{\mathcal{G}}.$$

Since $\mathbf{B} \setminus \tilde{\mathcal{G}}$ is projective over \tilde{X} , we can extend s' to $s'' : \tilde{V} \rightarrow \mathbf{B} \setminus \tilde{\mathcal{G}}$ such that $\tilde{Z} := \tilde{X} - \tilde{V}$ is of codimension at least two in \tilde{X} . Set $V^0 := X^0 - \varphi(\tilde{Z})$. Replacing \tilde{V} by $\varphi^{-1}(V^0)$, we may assume that \tilde{V} is a preimage of an open subscheme in X^0 .

Note that the two pullbacks of s'' to $\tilde{V} \times_{X^0} \tilde{V}$ coincide because they coincide over $\tilde{U} \times_{X^0} \tilde{U}$. This shows that s'' descends to a morphism $s^0 : V^0 \rightarrow \mathbf{B} \setminus \tilde{\mathcal{G}}$. Let \mathcal{B}^0 be the pullback of the \mathbf{B} -bundle $\tilde{\mathcal{G}} \rightarrow \mathbf{B} \setminus \tilde{\mathcal{G}}$ via s^0 , so that \mathcal{B}^0 is a \mathbf{B} -bundle over V^0 .

The isomorphism (1) gives rise to an isomorphism $\mathbf{G} \times^{\mathbf{B}} \mathcal{B}^0 \simeq \mathcal{G}^0|_{V^0}$, so that \mathcal{B}^0 is a \mathbf{B} -reduction of \mathcal{G}^0 over V^0 . \square

By Proposition 4.6, there is an open subscheme $X' \subset X^0$ containing x , and a quasi-elementary fibration $p : X' \rightarrow S$ with S connected and smooth over $\mathrm{Spec} B$ such that $Z \cap X'$ is finite over S . We may assume that S is affine. We will use the notation from Definition 3.1. In particular, we have a flat projective morphism $\bar{p} : \bar{X} \rightarrow S$. Set $s := \bar{p}(x)$, $F := \bar{p}^{-1}(s)$.

Note that $Z \cap X'$ is closed in \bar{X} (being finite over S), so applying Lemma 4.1(b) to $Z \cap X', Y \subset \bar{X}$, we find $Z_1 \subset X'$ such that Z_1 is a divisor in X' , $Z \cap X' \subset Z_1$, and Z_1 is finite over S (we might need to shrink (S, s)). We may assume that $\bar{X} - Z_1$ is an affine scheme. Then $X' - Z_1 = (\bar{X} - Z_1) \cap X'$ is also affine as the intersection of two open affine subschemes of a separated scheme.

Set $\mathcal{F} := \mathcal{G}^0|_{X'}$. Note that \mathcal{F} is reduced to the Borel subgroup \mathbf{B} over $X' - Z_1$, and $X' - Z_1$ is an affine scheme. Thus \mathcal{F} can be reduced to the torus \mathbf{T} on $X' - Z_1$. Indeed, it follows easily from [DG, Exp. XXII, Prop. 5.5.1] that there is a sequence of $\mathrm{Spec} B$ -group schemes

$$\mathbf{B} = \mathbf{B}_N \supset \dots \supset \mathbf{B}_1 \supset \mathbf{B}_0 = \mathbf{T}$$

such that for $i = 1, \dots, N$ we have $\mathbf{B}_i / \mathbf{B}_{i-1} \approx \mathbb{G}_a$ is the additive group scheme over $\mathrm{Spec} B$. Thus for all i we have an exact sequence

$$H_{\mathrm{\acute{e}t}}^1(X' - Z_1, \mathbf{B}_{i-1}) \rightarrow H_{\mathrm{\acute{e}t}}^1(X' - Z_1, \mathbf{B}_i) \rightarrow H_{\mathrm{\acute{e}t}}^1(X' - Z_1, \mathcal{O}_{X' - Z_1}).$$

By affineness of $X' - Z_1$ the last group is zero and the statement follows.

We claim that (after shrinking (S, s) again) we can find a divisor $Z_2 \subset X' - Z_1$ such that Z_2 is finite over S and \mathcal{F} is trivial over $X' - Z_1 - Z_2$. Since a principal bundle for a split torus is nothing but a collection of line bundles, this follows from the next lemma.

Lemma 4.10. *Let ℓ be a line bundle over $X'' := X' - Z_1$. Then (after shrinking (S, s)) there is a subscheme $Z'' \subset X''$ finite over S such that ℓ is trivial over $X'' - Z''$.*

Proof. First of all, we may extend ℓ to \bar{X} because \bar{X} is a regular scheme. Set $X_\infty := (\bar{X} - X'') \cap F$, this is a finite scheme. Adding finitely many points to X_∞ , we may assume that it intersects each irreducible component of F . Let A be the semilocal ring of X_∞ in \bar{X} . Since A is regular, ℓ is trivial over A . Thus there is a closed subscheme $Z'' \subset \bar{X}$ such that $\ell|_{\bar{X} - Z''}$ is trivial and $Z'' \cap X_\infty = \emptyset$. In particular, $Z'' \cap F$ is finite. Shrinking (S, s) we may assume that Z'' is finite over S and that $Z'' \subset X''$. \square

Note that $Z_1 \cup Z_2$ is closed in \bar{X} . By Lemma 4.1(c), shrinking (S, s) , we can find a finite surjective morphism $\Pi : \bar{X} \rightarrow \mathbb{P}_S^1$ such that

$$Z_1 \cup Z_2 \cup \{x\} \subset Z'' := \Pi^{-1}(0 \times S), \quad Y \subset Y' := \Pi^{-1}(\infty \times S).$$

Clearly, $X''' := \bar{X} - Y'$ is smooth and affine over S . Also, Z' is finite over S . It is easy to check that the restriction of p to X''' is a quasi-elementary fibration. Next, Z' is a principal divisor in X''' because $0 \times S$ is a principal divisor in \mathbb{A}_S^1 . Clearly, \mathcal{F} is trivial over $X''' - Z'$. This completes the proof of Proposition 3.2. \square

5. NICE TRIPLES: PROOF OF PROPOSITION 3.5

We use the notation of Proposition 3.2. Set $\mathcal{X}' := X' \times_S U$, let $q'_U : \mathcal{X}' \rightarrow U$ be the projection. Let $g \in H^0(X', \mathcal{O}_{X'})$ be an equation of Z' , set $f' = p_1^*(g) \in H^0(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$. Let Δ be the composition

$$U \xrightarrow{\text{diag}} U \times_S U \xrightarrow{\text{can} \times \text{Id}_U} X' \times_S U = \mathcal{X}'.$$

Let \mathcal{X} be the connected component of \mathcal{X}' containing $\Delta(U)$. Then \mathcal{X} is irreducible because it is regular and connected. Since $p : X' \rightarrow S$ is flat (even smooth) of relative dimension one, q'_U is also so, and we see that every component of each fiber is one-dimensional. Next, $\Delta^*(f') = g \neq 0$. Now it is easy to see that $(q'_U|_{\mathcal{X}}, f'|_{\mathcal{X}}, \Delta)$ is a nice triple. Let \mathcal{E}' be the pullback of \mathcal{F} to \mathcal{X}' and \mathcal{E} be the restriction of \mathcal{E}' to \mathcal{X} . It is clear that \mathcal{E} satisfies the conditions of our proposition, so this completes the proof in the case of infinite field $k(x)$.

Let $k(x)$ be finite. Let \mathcal{T} be a finite subscheme of \mathcal{X} intersecting every component of \mathcal{X}_x . Set $\mathcal{Y} := \mathcal{Z} \cup \Delta(U) \cup \mathcal{T}$. Clearly, \mathcal{Y} is finite over U ; let $\{y_1, \dots, y_m\}$ be all its closed points; let $S = \text{Spec}(\mathcal{O}_{y_1, \dots, y_m})$ be the corresponding semilocal scheme. Clearly, Δ factors through S .

Lemma 5.1 ([Pan1], Lemma 5.3). *Let S be a regular semilocal scheme over U ; let $\Delta : U \rightarrow S$ be a section. Then there exists a finite étale morphism $\rho : S' \rightarrow S$ and a section $\Delta' : U \rightarrow S'$ such that $\rho \circ \Delta' = \Delta$, $\Delta'(x)$ is the only $k(x)$ -rational point of the fiber S'_x , and for any integer $r \geq 1$ one has*

$$\#\{z \in S'_x \mid \deg[z : x] = r\} \leq \#\{z \in \mathbb{A}_x^1 \mid \deg[z : x] = r\}.$$

Proof. Let $S = \text{Spec } A$, let I be the ideal of $\Delta(U)$, so that $A = I \oplus R$. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ be all the maximal ideals of A . We may assume that \mathfrak{m}_1 is the ideal of $\Delta(x)$, that is, $\mathfrak{m}_1 \supset I$.

Choose a large number $N > 0$ and for each $i = 2, \dots, n$ a monic polynomial $f_i \in (A/\mathfrak{m}_i)[t]$ of degree N and such that

- if A/\mathfrak{m}_i is finite, then f_i is irreducible;

- if A/\mathfrak{m}_i is infinite, then f_i is a product of distinct monic polynomials of degree one.

Take $f_1 \in (A/\mathfrak{m}_1)[t]$ of the form tg , where g is irreducible of degree $N - 1$. By Chinese Remainder Theorem applied coefficientwise we can find a monic polynomial $f \in A[t]$ such that $\deg f = N$, $f \in I + tA[t]$ and $f \bmod \mathfrak{m}_i = f_i$ for all i . Set $S' = \text{Spec } A[t]/(f)$. Clearly, S' is finite and flat over S . The morphism Δ' is induced by the composition

$$A[t]/(f) \rightarrow A[t]/(I + tA[t]) = R.$$

We have

$$\#\{z \in S'_x \mid \deg[z : x] = r\} \begin{cases} = 1 \text{ if } r = 1, \\ = 0 \text{ if } 2 \leq r \leq N - 2, \\ \leq n \text{ if } r \geq N - 1. \end{cases}$$

Indeed, for every $i > 1$ such that A/\mathfrak{m}_i is finite, there is only one point of S'_x lying over \mathfrak{m}_i . On the other hand, if a point of S'_x lies over \mathfrak{m}_i such that A/\mathfrak{m}_i is infinite, then the degree of this point over x is infinite as well.

It is now easy to check that S' and Δ' satisfy our condition for N large enough. \square

Take ρ , S' and Δ' as in the above lemma. Clearing denominators, we can extend ρ and S' to a neighborhood of S to get a diagram

$$\begin{array}{ccccc} & S' & \hookrightarrow & \mathcal{V}' & \\ \Delta' \nearrow & \downarrow \rho & & \downarrow \theta & \\ U & \xrightarrow{\Delta} & S & \hookrightarrow & \mathcal{V} \hookrightarrow \mathcal{X}, \end{array}$$

where \mathcal{V} is an open subscheme of \mathcal{X} , θ is finite étale.

Note that $S \subset \mathcal{V}$ implies that $\mathcal{Y} \subset \mathcal{V}$ by the definition of S . The following lemma is similar to [Pan1, Lemma 5.4].

Lemma 5.2. *There is an open subscheme $\mathcal{W} \subset \mathcal{V}$ such that $\mathcal{W} \supset \mathcal{Y}$ and \mathcal{W} admits a finite U -morphism to \mathbb{A}_U^1 .*

Proof. We have a dominant morphism $\mathcal{X} \rightarrow \mathbb{A}_U^1$, which gives an embedding of the field of functions of \mathbb{A}_U^1 into the field of functions of \mathcal{X} . Let $\bar{\mathcal{X}}$ be the normalization of \mathbb{P}_U^1 in the field of functions of \mathcal{X} . Note that U is excellent and therefore Nagata ring, so normalization gives a finite morphism $\tilde{\Pi} : \bar{\mathcal{X}} \rightarrow \mathbb{P}_U^1$. Since \mathcal{X} is normal, $\tilde{\Pi}^{-1}(\mathbb{A}_U^1) = \mathcal{X}$. Thus $\bar{\mathcal{X}} - \mathcal{X}$ is finite over $\infty \times U$ and thus over U . Next, $\bar{\mathcal{X}}_x - \mathcal{V}_x = (\bar{\mathcal{X}}_x - \mathcal{X}_x) \cup (\mathcal{X}_x - \mathcal{V}_x)$ is finite (the second term is finite because it does not intersect \mathcal{T}_x). It follows that $\bar{\mathcal{X}} - \mathcal{V}$ is finite over U (indeed, it is projective and the closed fiber is finite). Using Lemma 4.1(c), we find a finite morphism $\bar{\Pi} : \bar{\mathcal{X}} \rightarrow \mathbb{P}_U^1$ such that $\bar{\Pi}(\mathcal{Y}) \subset 0 \times U$ and $\bar{\Pi}(\bar{\mathcal{X}} - \mathcal{V}) \subset \infty \times U$. It remains to take $\mathcal{W} := \bar{\Pi}^{-1}(\mathbb{A}_U^1)$. \square

Let \mathcal{W} be as in the above lemma. Let \mathcal{X}'' be the connected component of $\theta^{-1}(\mathcal{W})$ containing $\Delta'(U)$. Set $q_U'' := q_U \circ \theta|_{\mathcal{X}''}$ and $f'' = f \circ \theta|_{\mathcal{X}''}$. Then $(q_U'' : \mathcal{X}'' \rightarrow U, f'', \Delta')$ is the sought-for nice triple. The proof of Proposition 3.5 is complete.

6. BUNDLES OVER \mathbb{A}^1 : PROOF OF THEOREM 2

6.1. Horrocks type statement. Let R be a Noetherian local ring, $U := \text{Spec } R$. Let $x \in U$ be the closed point. Let \mathbf{G} be a split reductive group scheme over U . The following statement and its proof are close to [PSV, Thm. 9.6].

Proposition 6.1. *Let \mathcal{G} be a \mathbf{G} -bundle over \mathbb{P}_U^1 such that its restriction to \mathbb{P}_x^1 is a trivial \mathbf{G}_x -bundle. Then \mathcal{G} is isomorphic to the pullback of a \mathbf{G} -bundle over U .*

Proof. Consider a closed embedding $\mathbf{G} \rightarrow \mathbf{GL}(n, U)$ (existing e.g. by [Tho, Cor. 3.2]). Then by [KM, Cor. 1.2] the quotient $X := \mathbf{GL}(n, U)/\mathbf{G}$ exists as a separated algebraic space.

Consider the associated $\mathbf{GL}(n, U)$ -bundle $\mathcal{G}' := \mathbf{GL}(n, U) \times^{\mathbf{G}} \mathcal{G}$. Let, under the equivalence between $\mathbf{GL}(n, U)$ -bundles and rank n locally free sheaves, \mathcal{G}' correspond to the sheaf \mathcal{F} . Then \mathcal{F}_x is trivial, so according to [Gro3, Cor. 4.6.4], \mathcal{F} is trivial. Thus \mathcal{G}' is trivial as well.

Consider the morphism of exact sequences, induced by the canonical projection $pr_U : \mathbb{P}_U^1 \rightarrow U$,

$$\begin{array}{ccccc} \mathrm{Mor}_U(U, X) & \xrightarrow{\partial} & H_{\text{ét}}^1(U, \mathbf{G}) & \longrightarrow & H_{\text{ét}}^1(U, \mathbf{GL}(n, U)) \\ pr_U^* \downarrow & & \downarrow & & \downarrow \\ \mathrm{Mor}_U(\mathbb{P}_U^1, X) & \xrightarrow{\partial} & H_{\text{ét}}^1(\mathbb{P}_U^1, \mathbf{G}) & \longrightarrow & H_{\text{ét}}^1(\mathbb{P}_U^1, \mathbf{GL}(n, U)). \end{array}$$

The class of $[\mathcal{G}] \in H_{\text{ét}}^1(\mathbb{P}_U^1, \mathbf{G})$ is in the image of ∂ , because \mathcal{G}' is trivial. It remains to show that the morphism pr_U^* is surjective. Let $\iota_U : U \rightarrow \mathbb{P}_U^1$ be the embedding of the zero section. We will show that the composition

$$pr_U^* \circ \iota_U^* : \mathrm{Mor}_U(\mathbb{P}_U^1, X) \rightarrow \mathrm{Mor}_U(\mathbb{P}_U^1, X)$$

is the identity map. Let ω be the generic point of U . Since X is separated, it is enough to show that the base-changed morphism

$$pr_\omega^* \circ \iota_\omega^* : \mathrm{Mor}_\omega(\mathbb{P}_\omega^1, X_\omega) \rightarrow \mathrm{Mor}_\omega(\mathbb{P}_\omega^1, X_\omega)$$

is the identity map. However, \mathbb{P}_ω^1 is a projective scheme, while X_ω is an affine scheme by results of Haboush [Hab] and Nagata [Nag] (see also [Nis1, Corollary]). We see that every morphism from \mathbb{P}_ω^1 to X_ω factors through pr_ω and the proposition follows. \square

Remark 6.2. A reference to [KM] can be avoided as follows. View $X = \mathbf{GL}(n, U)/\mathbf{G}$ just as a space, that is, a sheaf on the big étale site $\mathrm{Aff}_{\text{ét}}$. We only need to show that two morphisms from $Y := \mathbb{P}_U^1$ to X coincide, provided they coincide at the generic point. By definition of the quotient, such morphisms can be lifted to morphisms $\tilde{Y} \rightarrow \mathbf{GL}(n, U)$, where $\tilde{Y} \rightarrow Y$ is an étale cover. Let $\varphi : \tilde{Y} \rightarrow \mathbf{GL}(n, U)$ be the quotient of these morphisms (with respect to the group structure on $\mathbf{GL}(n, U)$). By assumption, φ factors through \mathbf{G} at every generic point of \tilde{Y} . Thus φ factors through \mathbf{G} because \mathbf{G} is closed in $\mathbf{GL}(n, U)$.

6.2. Gluing principal bundles. Let $Y = 0 \times U$ be the zero divisor in \mathbb{P}_U^1 . Let $D_Y := \mathrm{Spec} R[[t]]$ be the “formal disc over Y ”, let $\dot{D}_Y := \mathrm{Spec} R((t))$ be the “punctured formal disc”. In [Fed, Sect. 4] we constructed a commutative diagram of morphisms of U -schemes

$$\begin{array}{ccc} \dot{D}_Y & \longrightarrow & D_Y \\ \downarrow & & \downarrow \\ \mathbb{P}_U^1 - Y & \longrightarrow & \mathbb{P}_U^1. \end{array}$$

Further, we explained that given a \mathbf{G} -bundle over $\mathbb{P}_U^1 - Y$, a \mathbf{G} -bundle over D_Y , and an isomorphism between their restrictions to \dot{D}_Y , we can glue the bundles to make a \mathbf{G} -bundle over \mathbb{P}_U^1 ; see [Fed, Prop. 4.4].

In particular, given a \mathbf{G} -bundle \mathcal{G} over \mathbb{P}_U^1 , its trivialization over \dot{D}_Y , and a loop $\alpha \in \mathbf{G}(R((t)))$, we can construct a new \mathbf{G} -bundle $\mathcal{G}(\alpha)$ over \mathbb{P}_U^1 as follows. We view α as an isomorphism between $\mathcal{G}|_{\dot{D}_Y}$ and the trivial \mathbf{G} -bundle over \dot{D}_Y , and use it to glue $\mathcal{G}|_{\mathbb{P}_U^1 - Y}$ with the trivial \mathbf{G} -bundle over D_Y .

6.3. Proof of Theorem 2. As before, let $Y = 0 \times U \subset \mathbb{P}_U^1$. Since $\mathbb{P}_U^1 - Y \simeq \mathbb{A}_U^1$, we may view \mathcal{F} as a \mathbf{G} -bundle over $\mathbb{P}_U^1 - Y$. Let us trivialize \mathcal{F} on a complement of a subscheme $Z \subset \mathbb{P}_U^1 - Y$ finite over U . Note that Z is closed in \mathbb{P}_U^1 . Let us extend \mathcal{F} to a \mathbf{G} -bundle $\tilde{\mathcal{F}}$ over \mathbb{P}_U^1 by gluing \mathcal{F} with the trivial bundle over $\mathbb{P}_U^1 - Z$ (observe that both bundles are trivial over the intersection $\mathbb{P}_U^1 - Y - Z$).

Consider the \mathbf{G}_x -bundle $\tilde{\mathcal{F}}_x$ over \mathbb{P}_x^1 obtained by restricting $\tilde{\mathcal{F}}$. Note that $\tilde{\mathcal{F}}_x$ is generically trivial because it is trivial over $\mathbb{P}_x^1 - Z_x$. Thus it is trivial over $\mathbb{P}_x^1 - 0$ by [Gil1, Cor. 3.10(a)]. Fix such a trivialization.

On the other hand, $\tilde{\mathcal{F}}$ is trivialized over D_Y , as the morphism $D_Y \rightarrow \mathbb{P}_U^1$ factors through $\mathbb{P}_U^1 - Z$. Fix such a trivialization, it gives rise to a trivialization of $\tilde{\mathcal{F}}_x$ over D_{Y_x} .

Thus we get two trivializations of $\tilde{\mathcal{F}}_x$ over \dot{D}_{Y_x} ; they differ by an element

$$\alpha \in \mathbf{G}(\dot{D}_{Y_x}) = \mathbf{G}(k((t))),$$

where $k := k(x)$.

Lemma 6.3. *There is $\tilde{\alpha} \in \mathbf{G}(R((t)))$ extending α .*

Proof. Let \mathbf{T} be a split maximal torus in \mathbf{G} . Let \mathbf{B} be a Borel subgroup scheme such that $\mathbf{T} \subset \mathbf{B} \subset \mathbf{G}$. Let \mathbf{B}^- be the opposite Borel subgroup scheme (see [DG, Exp. XXII, Prop. 5.9.2]). Let \mathbf{U}^- and \mathbf{U} be the unipotent radicals of \mathbf{B}^- and \mathbf{B} respectively. Let E be the subgroup of the abstract group $\mathbf{G}(k((t)))$ generated by $\mathbf{U}_-(k((t)))$ and $\mathbf{U}(k((t)))$. Combining [Tit, Sect. 3, (17) and (18)] and [Gil2, Fait 4.3] we get $\mathbf{G}(k((t))) = \mathbf{T}(k((t))) \cdot E$.

Next, every element of E extends to $\mathbf{G}(R((t)))$, see [FP, Lemma 5.24]. Thus, it remains to show that every element of $\mathbf{T}(k((t)))$ extends to $\mathbf{T}(R((t)))$. Since \mathbf{T} is split, it is enough to show that every non-zero element of $k((t))$ extends to an invertible element of $R((t))$, which is obvious because R is local. \square

Since $\tilde{\mathcal{F}}$ is trivialized over \dot{D}_Y and $\tilde{\alpha}^{-1} \in \mathbf{G}(R((t)))$, we obtain a new principal bundle $\tilde{\mathcal{F}}(\tilde{\alpha}^{-1})$ over \mathbb{P}_U^1 (see the end of Section 6.2).

It is easy to see from the construction, that the restriction of $\tilde{\mathcal{F}}(\tilde{\alpha}^{-1})$ to \mathbb{P}_x^1 is a trivial \mathbf{G}_x -bundle. By Proposition 6.1, $\tilde{\mathcal{F}}(\tilde{\alpha}^{-1})$ is isomorphic to a pullback of a \mathbf{G} -bundle over U . Since the restriction of $\tilde{\mathcal{F}}(\tilde{\alpha}^{-1})$ to $Y = 0 \times U$ is trivial, we see that $\tilde{\mathcal{F}}(\tilde{\alpha}^{-1})$ is trivial. Finally, we see that

$$\mathcal{F} \simeq \tilde{\mathcal{F}}(\tilde{\alpha}^{-1})|_{\mathbb{P}_U^1 - (0 \times U)}$$

is trivial. The proof of theorem 2 is complete. \square

REFERENCES

- [CTO] Jean-Louis Colliot-Thélène and Manuel Ojanguren. Espaces principaux homogènes localement triviaux. *Inst. Hautes Études Sci. Publ. Math.*, (75):97–122, 1992.
- [CTS] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc. Principal homogeneous spaces under flasque tori: applications. *J. Algebra*, 106(1):148–205, 1987.

- [DG] Michel Demazure and Alexander Grothendieck. *Schémas en groupes. III: Structure des schémas en groupes réductifs*. Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 153. Springer-Verlag, Berlin, 1970.
- [Fed] Roman Fedorov. Affine Grassmannians of group schemes and exotic principal bundles over \mathbb{A}^1 . *Amer. Journal of Math.*, 138(4):879–906, 2016.
- [FP] Roman Fedorov and Ivan Panin. A proof of the Grothendieck–Serre conjecture on principal bundles over regular local rings containing infinite fields. *Publications mathématiques de l’IHÉS*, 122(1):169–193, 2015.
- [Gille1] Philippe Gille. Torseurs sur la droite affine. *Transform. Groups*, 7(3):231–245, 2002.
- [Gille2] Philippe Gille. Le problème de Kneser-Tits. *Astérisque*, (326):Exp. No. 983, vii, 39–81 (2010), 2009. Séminaire Bourbaki. Vol. 2007/2008.
- [Gro1] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, (32):361, 1967.
- [Gro2] Alexander Grothendieck. Torsion homologique et sections rationnelles. In *Anneaux de Chow et applications, Séminaire Claude Chevalley*, number 3. Paris, 1958.
- [Gro3] Alexander Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. *Inst. Hautes Études Sci. Publ. Math.*, (11):167, 1961.
- [Gro4] Alexander Grothendieck. Éléments de géométrie algébrique. IV. Étude local des schémas et des morphismes des schémas, seconde partie. *Inst. Hautes Études Sci. Publ. Math.*, (24):5–231, 1965.
- [Gro5] Alexander Grothendieck. Le groupe de Brauer. II. Théorie cohomologique. In *Dix Exposés sur la Cohomologie des Schémas*, pages 67–87. North-Holland, Amsterdam, 1968.
- [Gro6] Alexander Grothendieck. Technique de descente et théorèmes d’existence en géométrie algébrique. I. Généralités. Descente par morphismes fidèlement plats. In *Séminaire Bourbaki, Vol. 5, Exp. No. 190.*, pages 299–327. Soc. Math. France, Paris, 1995.
- [Hab] William J. Haboush. Reductive groups are geometrically reductive. *Ann. of Math.* (2), 102(1):67–83, 1975.
- [KM] Seán Keel and Shigefumi Mori. Quotients by groupoids. *Ann. of Math.* (2), 145(1):193–213, 1997.
- [Knu] Max-Albert Knus. *Quadratic and Hermitian forms over rings*, volume 294 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1991. With a foreword by I. Bertuchini.
- [Mat] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.
- [Nag] Masayoshi Nagata. Invariants of a group in an affine ring. *J. Math. Kyoto Univ.*, 3:369–377, 1963/1964.
- [Nis1] Yevsey Nisnevich. Affine homogeneous spaces and finite subgroups of arithmetic groups over function fields. *Functional Analysis and Its Applications*, 11(1):64–66, 1977.
- [Nis2] Yevsey Nisnevich. Espaces homogènes principaux rationnellement triviaux et arithmétique des schémas en groupes réductifs sur les anneaux de Dedekind. *C. R. Acad. Sci. Paris Sér. I Math.*, 299(1):5–8, 1984.
- [Nis3] Yevsey Nisnevich. Rationally trivial principal homogeneous spaces, purity and arithmetic of reductive group schemes over extensions of two-dimensional regular local rings. *C. R. Acad. Sci. Paris Sér. I Math.*, 309(10):651–655, 1989.
- [Pan1] Ivan Panin. On Grothendieck–Serre conjecture concerning principal G-bundles over regular semi-local domains containing a finite field: I. *ArXiv e-prints*, June 2014.
- [Pan2] Ivan Panin. On Grothendieck–Serre conjecture concerning principal G-bundles over regular semi-local domains containing a finite field: II. *ArXiv e-prints*, June 2014.
- [Pan3] Ivan Panin. Proof of Grothendieck–Serre conjecture on principal G-bundles over regular local rings containing a finite field. *ArXiv e-prints*, June 2014.
- [Pan4] Ivan Panin. Proof of Grothendieck–Serre conjecture on principal bundles over regular local rings containing a finite field. <https://www.math.uni-bielefeld.de/LAG/man/559.html>, September 2015.
- [Po] Bjorn Poonen. Bertini theorems over finite fields. *Ann. of Math.* (2), 160(3):1099–1127, 2004.
- [Pop] Dorin Popescu. General Néron desingularization and approximation. *Nagoya Math. J.*, 104:85–115, 1986.

- [PS] Ivan Panin and Anastasia Stavrova. On the Grothendieck–Serre conjecture concerning principal G -bundles over semi-local Dedekind domains. *ArXiv e-prints*, 1512.00354, 2015.
- [PSV] I. Panin, A. Stavrova, and N. Vavilov. On Grothendieck–Serre’s conjecture concerning principal G -bundles over reductive group schemes: I. *Compos. Math.*, 151(3):535–567, 2015.
- [Ser] Jean-Pierre Serre. Espaces fibrés algébrique. In *Anneaux de Chow et applications, Séminaire Claude Chevalley*, number 3. Paris, 1958.
- [SGA] *Théorie des topos et cohomologie étale des schémas. Tome 3*. Lecture Notes in Mathematics, Vol. 305. Springer-Verlag, Berlin, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat.
- [Spi] Mark Spivakovsky. A new proof of D. Popescu’s theorem on smoothing of ring homomorphisms. *J. Amer. Math. Soc.*, 12(2):381–444, 1999.
- [Swan] Richard G. Swan. Néron–Popescu desingularization. In *Algebra and geometry (Taipei, 1995)*, volume 2 of *Lect. Algebra Geom.*, pages 135–192. Int. Press, Cambridge, MA, 1998.
- [Tho] R. W. Thomason. Equivariant resolution, linearization, and Hilbert’s fourteenth problem over arbitrary base schemes. *Adv. in Math.*, 65(1):16–34, 1987.
- [Tit] J. Tits. Algebraic and abstract simple groups. *Ann. of Math. (2)*, 80:313–329, 1964.

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